Some sets defined by digits. Furstenberg Lemma

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Furstenberg lemma. Let $b \in \mathbb{N}$, $b > 2$, and

$T_b(x) : [0,1] \to [0,1]$, $x \mapsto bx \pmod 1$ – b-adic shift.

Assume $K \subset [0,1]$ – T_b-invariant compact ($T_b(K) = K$). Then $\mathrm{Hdim}\, K = \mathrm{Mdim}\, K$

Proof. We always have $\mathrm{Hdim}\, K \le \overline{\mathrm{Mdim}}\, K$, so let us prove the opposite.

Encode every b-adic interval by b-multiindex (and every $x \in [0,1]$ by sequence $(x_1, \dots, x_n, \dots)$, $x_j \in \{0, \dots, b-1\}$ via $x = \sum x_j b^{-j}$).

Fix $\alpha > \mathrm{Hdim}\, K$. $\exists$ covering (K_i) of K by intervals with $\sum (\mathrm{diam}\, K_i)^\alpha < \frac{1}{2(b+1)}$. Each K_i can be covered by at most $(b+1)$ b-adic intervals of diam $\le \mathrm{diam}\, K_i$, so there is a covering of K by b-adic intervals I_{σ_i} with $\underset{\sum b^{-\alpha\, \mathrm{length}(\sigma_i)}}{\sum (\mathrm{diam}\, I_{\sigma_i})^\alpha} < (b+1)^\alpha \cdot \frac{1}{2(b+1)} < \frac{1}{2}$.

Let $\Sigma = \{\sigma_i\}$ – collection of the multi-indices from this covering, $M = \max_{\sigma \in \Sigma}(\mathrm{length}\, \sigma)$.

Assume now that $\mathrm{length}\, \sigma > M$ and $I_\sigma \cap K \ne \emptyset$. Then, since $T_b K = K$, we can apply shift to σ till we get an element of Σ! (since they cover K). Thus $\sigma = \tau_1 \sigma_1$, where $\sigma_1 \in \Sigma$.

If $\mathrm{length}\, \tau_1 > M$, we can continue to get $\tau_1 = \tau_2 \sigma_2$, so $\sigma = \tau_2 \sigma_2 \sigma_1$.

Continuing with this logic, we get $\sigma = \tau \sigma_k \dots \sigma_1$ for any σ with $\sigma_j \in \Sigma$, and $\mathrm{length}\, \tau \le M$ (thus, there are only b^M choices of τ).

Now let us observe that

$$\circledast = \sum_{\sigma : I_\sigma \cap K \ne \emptyset} b^{-\alpha\, \mathrm{length}\, \sigma} = \sum_{\tau,\, \mathrm{length}\, \tau \le M} b^{-\alpha\, \mathrm{length}\, \tau} \sum_{n=0}^{\infty} \sum_{\sigma_1 \dots \sigma_n \in \Sigma} b^{-\alpha\, \mathrm{length}(\sigma_1 \dots \sigma_n)} \le b^M \sum_{n=0}^{\infty} \left(\sum b^{-\alpha\, \mathrm{length}\, \sigma_i}\right)^n \le \frac{b^M}{1 - \frac{1}{2}}$$

$$= 2b^M < \infty.$$

By note that $\#\{\sigma : I_\sigma \cap K \ne \emptyset,\ \mathrm{length}\, \sigma = n\} \ge N(K, b^{-n})$.

Thus $\circledast \ge \sum b^{-\alpha n} N(K, b^{-n})$, which means that $b^{-\alpha n} N(K, b^{-n}) \to 0$, so $\underset{\overline{\mathrm{Mdim}}\, K}{\overline{\lim}} \frac{\log N(K, \varepsilon)}{|\log \varepsilon|} \le \overline{\lim} \frac{\log N(K, b^{-n})}{\log b^{-n+1}} \le \alpha$. ∎

As an immediate application, we see that, since $T_3 C = C$, $\mathrm{Hdim}\, C = \mathrm{Mdim}\, C = \frac{\log 2}{\log 3}$

Let us now consider certain ~~sets~~ defined by digits.

Let A be a $b \times b$ matrix with $\{0,1\}$ entries.

$x \in C_A$ if $x = \sum x_j b^{-j}$ (b-adic form) and $\forall j$ $A_{x_j x_{j+1}} = 1$.

so $C = C_A$ for $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

We also assume that if j-th row of matrix is identically 0, so is j-th column – to allow continuation of any sequence not started with j;

C_A is T_b-invariant, so, by Furstenberg, $\mathrm{Hdim}\, C_A = \mathrm{Mdim}\, C_A$.

Let us compute $\mathrm{Mdim}\, C_A$.

We can simply count $N_n = \#\{\sigma : I_\sigma \cap C_A \ne \emptyset,\ \mathrm{length}\, \sigma = n\}$,

This is the same as $\#\{(x_1 \dots x_n) : A_{x_i x_{i+1}} = 1\ \forall i < n\}$.

Note that $A^n_{x_1 x_n} = \#\{\sigma : \sigma_1 = x_1, \sigma_n = x_n, A_{\sigma_i \sigma_{i+1}} = 1\}$.

so $N_n = \sum_{i=1}^{b} \sum_{j=1}^{b} A^n_{ij}$. And $\mathrm{Mdim}\, C_A = \lim_{n \to \infty} \frac{\log N_n}{n \log b} = \frac{\log \lim_{n \to \infty} N_n^{1/n}}{\log b} = \frac{\log \rho(A)}{\log b}$, where

$\rho(A) = \lim_{n \to \infty} \left(\sum_{i=1}^{b} \sum_{j=1}^{b} A^n_{ij}\right)^{1/n}$ – spectral radius of A.